# Random Walks on Density-Ramanujan Complexes

by

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# Abstract

Let $\mathbf{G}$ be a semisimple group over a non-archimedean local field $F$, and let $\mathcal{B}$ be the Bruhat-Tits building of $G = \mathbf{G}(F)$. In [LLP20], it was shown that if $T$ is a $G$-equivariant, regular, and collision-free branching operator on a set of cells $\mathfrak{C} \subset \mathcal{B}$, then the induced walks on Ramanujan quotients $\Gamma \backslash \mathcal{B}$ exhibit the cutoff phenomenon. For $\mathbf{G} = \mathrm{PGL}_d$, simple random walks on Ramanujan complexes were also shown to exhibit cutoff in [CP22]. In this dissertation, we establish cutoff for these walks under the weaker assumption that the complexes form an expander family and satisfy Sarnak's density hypothesis.

# Dedications

To my parents, Abdellah and Fadma Idelhaj, my aunt Tita, and to all my family.

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# Chapter 1

# Introduction

## 1.1 The Cutoff Phenomenon

Consider a finite, ergodic Markov chain on a state space $\mathfrak{X}$ with transition kernel $P$. For probability distributions $\mu$ and $\nu$ on $\mathfrak{X}$, we consider the total variation norm:

$$\|\mu - \nu\|_{\mathrm{TV}} = \max_{A \subset \mathfrak{X}} |\mu(A) - \nu(A)| = \frac{1}{2}\|\mu - \nu\|_1.$$

We define the $\varepsilon$-mixing time of the chain to be the smallest $t$ after which the time $t$ distribution $\mu_x^t(\cdot) = P^t(x, \cdot)$ becomes $\varepsilon$-close to the stationary distribution $u$:

$$t_{\mathrm{mix}}(\varepsilon) = \min\left\{t \in \mathbb{N} \mid \max_{x \in \mathfrak{X}} \|\mu_x^t - u\|_{\mathrm{TV}} < \varepsilon\right\}.$$

Since advancing a Markov chain brings it closer to stationary in total variation, we have $\|\mu_x^t - u\|_{\mathrm{TV}} \leq \varepsilon$ for all $t \geq t_{\mathrm{mix}}(\varepsilon)$ and all $x \in \mathfrak{X}$, i.e. the walk stays $\varepsilon$-close to stationary at all further times.

**Definition 1.1.** A family of Markov chains $(\mathfrak{X}_N, P_N)$ exhibits (total variation) *cutoff* if

$$\lim_{N \to \infty} \frac{t_{\mathrm{mix}}(\delta, N)}{t_{\mathrm{mix}}(1 - \delta, N)} = 1,$$

for every $\delta > 0$. We say the cutoff happens at time $t(N)$ if, for every $\delta > 0$, there exists a window of size $w(N, \delta) = o(t(N))$ such that

$$|t_{\text{mix}}(\delta, N) - t(N)| < w(N, \delta)$$

for large enough $N$.

Informally, the distance between $\mu_x^t$ and $u$ starts off as large as possible until close to time $t(N)$, and then rapidly decreases to near 0 within the cutoff window.

The first studied instance of the cutoff phenomenon is in the setting of random transpositions in [DS81]. Consider the probability distribution $T$ on $S_n$ for which

- $T(e) = \frac{1}{n}$ where $e$ is the identity,
- $T(\tau) = \frac{2}{n^2}$ if $\tau$ is a transposition, and
- $T(\cdot) = 0$ otherwise.

The distribution $T$ is a model of a random transposition, and its $k$-fold convolution $T^{*k}$ is that of $k$ random transpositions. Let $U$ be the uniform distribution on $S_n$.

**Theorem 1.2.** *Random transpositions on $S_n$ exhibit cutoff at time $\frac{1}{2}n\log(n)$ with window $O(n)$.*

The term "cutoff phenomenon" first appeared in the work of Aldous and Diaconis [AD86], which showed that the top-in-at-random shuffle exhibits cutoff. In these and many other early demonstrations of the cutoff phenomenon, the Markov chains exhibit the property that the degrees of the underlying digraphs tend to infinity as the number of states does. We move to consider Markov chains in which the degrees are bounded as the number of states grows.

## 1.2 Expansion

We now move to consider the classes of bounded degree graphs expected to exhibit cutoff behavior, known as *expanders*. We consider a simple undirected graph $X = (V, E)$ with $n$ vertices and $m$ edges.

**Definition 1.3.** The Cheeger constant of $X$ is defined to be

$$h(X) = \min_{S \subset V} \frac{|E(S, S^c)|}{\min(|S|, |S^c|)},$$

where $E(S, S^c)$ is the set of edges with one endpoint in $S$ and the other in $S^c$.

Note that if $X$ is disconnected, then $h(X) = 0$, as we may choose $S$ to be a connected component.

**Definition 1.4.** A graph is an $\varepsilon$-expander if $h(X) \geq \varepsilon$. A family of graphs $\mathcal{F}$ of bounded degree is said to be an expander family if there exists some $\varepsilon > 0$ such that all $X \in \mathcal{F}$ are $\varepsilon$-expanders.

If $X$ is an $\varepsilon$-expander and $S$ is a set of vertices with $|S| \leq n/2$, then to disconnect $S$ from its complement, one must delete at least $\varepsilon|S|$ edges. From this, we see that expansion is a quantitative measure of "well-connectedness" of a graph. The assumption that the elements of an expander family have bounded degree implies that they are sparse: the number of edges grows as $O(n)$, as opposed to $O(n^2)$ for the family of complete graphs.

Intuitively, one expects expanders to have good mixing properties for simple random walk, due to the lack of bottlenecks. To make this intuition precise, we turn to spectral theory. Let $A_X : L^2(V) \to L^2(V)$ be the adjacency operator

$$(A_X(f))(x) = \sum_{y \sim x} f(y).$$

The following proposition collects standard facts about the spectrum of $A_X$.

**Proposition 1.5.** *Denote the spectrum of $A_X$ by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then the following are true:*

1. *All eigenvalues $\lambda$ of $A_X$ satisfy $|\lambda| \leq \max_{x \in X} \deg(x)$.*

2. *The largest eigenvalue $\lambda_1$ satisfies*

$$\lambda_1 \geq \frac{\sum_{x \in V} \deg(x)}{n}.$$

*In particular, if $X$ is $k$-regular, then $\lambda_1 = k$.*

3. *If $X$ is $k$-regular, then the $k$-eigenspace of $A_X$ is the space of locally constant functions on $X$. In particular, the multiplicity of $k$ as an eigenvalue of $A_X$ is the number of connected components of $X$.*

4. *If $X$ is bipartite, then the spectrum of $A_X$ is symmetric about 0, i.e. $\lambda_i = -\lambda_{n+1-i}$ for $1 \leq i \leq n$.*

5. *If $X$ is connected and $\lambda_n = -\lambda_1$, then $X$ is bipartite.*

For a $k$-regular[1] graph $X$, we define its spectral gap to be $\lambda_1 - \lambda_2 = k - \lambda_2$. By 3, the spectral gap is 0 if and only if $X$ is disconnected, a property shared by the Cheeger constant. A stronger relationship between these two quantities is furnished by the Cheeger-Buser inequality.

**Theorem 1.6.** *Let $X$ be a $k$-regular graph. Then*

$$\frac{k - \lambda_2(X)}{2} \leq h(X) \leq \sqrt{2k(k - \lambda_2(X))}.$$

A consequence of the Cheeger-Buser inequality is that a family $\mathcal{F}$ of $k$-regular graphs is an expander family if and only if the spectral gaps $k - \lambda_2(X)$ are bounded away from 0. From this, we may now make precise the intuition that expansion contributes to mixing of random walk.

**Proposition 1.7.** *Let $X$ be a $k$-regular graph with spectrum $k = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Set $|\lambda| = \max(|\lambda_2|, \ldots, |\lambda_n|)$. For a vertex $x \in V$, we let $\mu_x^t$ denote the time $t$ distribution for simple random walk on $X$ starting at $x$, and let $u$ denote the stationary distribution. Then*

$$\|\mu_x^t - u\|_{\mathrm{TV}} \leq \frac{\sqrt{n}}{2} \left(\frac{\lambda}{k}\right)^t.$$

[1] For irregular graphs, the spectral gap is best defined by considering the normalized Laplace operator, rather than the adjacency operator.

*Proof.* Since $X$ is $k$-regular, the time $t$ distribution is given by $\mu_x^t = \left(\frac{1}{k}A\right)^t \mathbb{1}_x$, and the stationary distribution is $u = \frac{1}{n}\mathbb{1}$. Note that

$$\left\langle \mathbb{1}_x - \frac{1}{n}\mathbb{1}, \mathbb{1} \right\rangle = 1 - \frac{1}{n} + \sum_{y \neq x} \left(-\frac{1}{n}\right) = 0,$$

so $\mathbb{1}_x - u \in \mathbb{1}^\perp$. The operator norm of $A$ on $\mathbb{1}^\perp$ is $\lambda$, by Courant-Fischer. We now use Cauchy-Schwarz and estimate

$$\begin{aligned}
\left\|\mu_x^t - u\right\|_{\mathrm{TV}} &\leq \frac{\sqrt{n}}{2} \left\|\mu_x^t - u\right\|_2 \\
&= \frac{\sqrt{n}}{2} \left\| \left(\frac{1}{k}A\right)^t (\mathbb{1}_x - u) \right\|_2 \\
&\leq \frac{\sqrt{n}}{2} \left\| \left(\frac{1}{k}A\right)^t \Big|_{\mathbb{1}^\perp} \right\| \cdot \left\| \mathbb{1}_x - \frac{1}{n}\mathbb{1} \right\|_2 \\
&\leq \frac{\sqrt{n}}{2} \left(\frac{\lambda}{k}\right)^t.
\end{aligned}$$

□

*Remark* 1.8. We note two geometric obstructions for $\mu_x^t$ to converge to stationary:

1. If $X$ is disconnected, then $\mu_x^t$ is supported on the connected component of $x$.
2. If $X$ is bipartite, then $\mu_x^t$ is supported on the side of the bipartition of $X$ containing $x$ if $t$ is even, and on the side not containing $x$ if $t$ is odd.

By Proposition 1.5, $\lambda < 1$ if and only if $X$ is connected and non-bipartite, and thus we have geometric convergence to stationary by Proposition 1.7.

Peres conjectured in 2004 that simple random walk on any family of $k$-regular transitive expanders exhibits cutoff[2]. We note the following theorem of Lubetzky and Sly in [LS10].

**Theorem 1.9.** *Fix $k \geq 3$. Then random $k$-regular graphs, with high probability, exhibit cutoff at time $\frac{k}{k-2}\log_{k-1}(n)$, with window $O(\sqrt{\log(n)})$.*

[2]Counterexamples are known in the absence of transitivity.

The idea of the proof is to consider the non-backtracking random walk (NBRW) on directed edges, defined by the operator

$$Bh(u \to v) = \frac{1}{k-1} \sum_{v \to w \neq u} h(v \to w).$$

This walk mixes faster than the simple random walk, and exhibits cutoff with a sharper window.

**Theorem 1.10.** *Fix $k \geq 3$. Then random $k$-regular graphs, with high probability, exhibit cutoff with respect to NBRW at time $\log_{k-1}(kn)$, with window of constant size.*

To reduce Theorem 1.9 to Theorem 1.10, Lubetzky and Sly use an observation of Peres that the distribution of a simple random walk conditioned on its lift to the tree having distance $t$ from the root is that of a time $t$ non-backtracking random walk[3]. They then show that the time it takes for SRW on the tree to reach distance $\log_{k-1}(n)$ from the root (at which time NBRW mixes) is concentrated around $\frac{k}{k-2}\log_{k-1}(kn)$, with standard deviation of order $O(\sqrt{\log(n)})$.

Since random regular graphs are good expanders with high probability, this provides some evidence toward Peres' conjecture. However, it remains open. We now turn to a family of graphs for which simple random walk is known to exhibit cutoff.

## 1.3 Ramanujan Graphs

**Definition 1.11.** A connected, $k$-regular graph $X$ is called *Ramanujan* if all of its adjacency eigenvalues $\lambda$ satisfy $|\lambda| = k$ or $|\lambda| \leq 2\sqrt{k-1}$.

Since $X$ is connected and $k$-regular, Proposition 1.5 tells us that $\lambda = k$ is an eigenvalue of multiplicity 1, and $-k$ is an eigenvalue if and only if $X$ is bipartite, in which case its multiplicity also equals 1. We call these the *trivial eigenvalues* of $A_X$.

*Remark* 1.12. By the work of Alon and Boppana, a family of $k$-regular graphs $X_N$ with order tending to infinity satisfies

$$\liminf_{N \to \infty} \lambda_2(X_N) \geq 2\sqrt{k-1}.$$

[3]While NBRW is a walk on directed edges, we may project the edge $u \to v$ to the vertex $v$ and obtain a distribution on vertices.

Thus, Ramanujan graphs have asymptotically optimal spectral gap. In addition, we note that the $k$-regular tree $T_k$ has spectral radius $2\sqrt{k-1}$. Thus, we may think of Ramanujan graphs as those whose non-trivial spectrum is contained in that of their universal cover.

Lubetzky and Peres in [LP16] show that simple random walk on Ramanujan graphs exhibits the cutoff phenomenon.

**Theorem 1.13.** *The family of $k$-regular non-bipartite Ramanujan graphs $X_N$ exhibits cutoff with respect to simple random walk at time $\frac{k}{k-2}\log_{k-1}(|X_N|)$ with window $O(\sqrt{\log(|X_N|)})$.*

The strategy of Lubetzky and Peres was to show that the operator $B$ associated to the NBRW on $X$ satisfies $|\lambda| \leq \sqrt{k-1}$ for any non-trivial eigenvalue $\lambda$, i.e. that the associated digraph is a *Ramanujan digraph*. In contrast to the SRW, the NBRW lifted to the universal cover creates no cycles, which contributes to the improved spectral bound (see Lemma 3.1). In addition, the operator is 2-*normal*: there exists an orthonormal basis of $L^2(\vec{E})$ in which the matrix of $B$ is block diagonal with blocks of size at most $2 \times 2$.

**Theorem 1.14.** *The family of $k$-regular Ramanujan graphs $X_N$ exhibits cutoff with respect to NBRW at time $\log_{k-1}(|X_N|)$ with window $O(\log_k(\log(|X_N|)))$.*

The reduction of Theorem 1.13 to Theorem 1.14 is similar to the analogous reduction in the case of random regular graphs.

## 1.4 The Density Hypothesis

Golubev and Kamber in [GK22] consider a weakening of the Ramanujan hypothesis which still suffices for exhibiting cutoff for random walk. This "density hypothesis" is based on a conjecture of Sarnak which substitutes for the Ramanujan conjecture in many applications. For now, we state the density hypothesis in the language of spectral graph theory, postponing a more general discussion until §2.1.

Let $X = (V, E)$ be a connected, $k$-regular graph. Its universal cover is the $k$-regular tree $T_k$, which

we take to be rooted at some point $x_0$. For a function $f : V \to \mathbb{C}$, we let $\widetilde{f} : T_k \to \mathbb{C}$ denote its universal cover, and define $\rho_{v_0}(\widetilde{f}) : V(T_k) \to \mathbb{C}$ by

$$\rho_{v_0}(\widetilde{f})(v) = \frac{1}{|S^d_{v_0}|} \sum_{v' \in S^d_{v_0}} \widetilde{f}(v'),$$

where $d$ is the distance from $v_0$ to $v$, and $S^d_{v_0}$ denotes the sphere of radius $d$ centered at $v_0$.

**Proposition 1.15.** *Let $X$ be a connected, $k$-regular, and non-bipartite graph with adjacency operator $A_X$. Then the following are equivalent:*

1. *Every non-trivial $\lambda$ of $A_X$ satisfies $|\lambda| \leq (k-1)^{1/p} + (k-1)^{1-1/p}$.*
2. *Every non-trivial eigenvalue $\lambda$ of the non-backtracking operator $B_X$ satisfies $|\lambda| \leq (k-1)^{1-1/p}$.*
3. *For any $f \in L^2(V)$ orthogonal to the constant function, any $v_0 \in T_k$, and any $\varepsilon > 0$, $\rho_{v_0}(\widetilde{f}) \in L^{p+\varepsilon}(T_k)$.*

*Remark* 1.16. The quantity $(k-1)^{1/p} + (k-1)^{1-1/p}$ is the operator norm of the adjacency operator $A_{T_k}$ acting on $L^p(T_k)$. When $p = 2$, the first condition in Proposition 1.15 is equivalent to being a Ramanujan graph.

**Definition 1.17.** Let $X = (V, E)$ be a connected, $k$-regular graph with adjacency operator $A_X$. For an eigenvalue $\lambda$ of $A_X$, we associate its $p$-value $p(\lambda)$ as follows:

- If $|\lambda| \leq 2\sqrt{k-1}$, we set $p(\lambda) = 2$.
- Otherwise, we choose $p = p(\lambda)$ such that:

$$|\lambda| = (k-1)^{1/p(\lambda)} + (k-1)^{1-1/p(\lambda)}.$$

We say $X$ is an $L^p$-expander if all non-trivial eigenvalues $\lambda$ of $A_X$ satisfy $p(\lambda) \leq p$.

Note that the usual definition of a family $\mathcal{F}$ of $k$-regular graphs being an expander family, via the Cheeger constant, is equivalent to each $X \in \mathcal{F}$ being an $L^p$ expander for some fixed $p$, by the

Cheeger-Buser inequality and Proposition 1.15. For more information about $L^p$-expansion in the setting of graphs, we encourage the reader to consult [K19], from which the preceding content is drawn.

**Definition 1.18.** A family $\mathcal{F}$ of graphs is called *density-Ramanujan* if, for every $X \in \mathcal{F}$, $p > 2$, and $\varepsilon > 0$, we have:

$$\#\{\lambda \in \mathrm{Spec}(A_X) \mid p(\lambda) \geq p\} \ll_\varepsilon |X|^{2/p+\varepsilon}.$$

Note that Ramanujan graphs satisfy $\#\{\lambda \in \mathrm{Spec}(A_X) \mid p(\lambda) \geq p\} \leq 2$ for $p > 2$, so being density-Ramanujan is weaker than being Ramanujan. Assuming this, Golubev and Kamber exhibit cutoff for simple random walk.

**Theorem 1.19.** *Let $\mathcal{F}$ be a family of finite, connected, non-bipartite, $k$-regular graphs. Then for any $\varepsilon_0, \varepsilon_1, \delta > 0$:*

1. *There exists $n_0$ depending on $\varepsilon_1, \delta$ such that for any $X \in \mathcal{F}$ with order $n > n_0$, any $x_0 \in X$, and $t < (1-\varepsilon_1)\frac{k}{k-2}\log_{k-1}(n)$, it holds that*

$$\left\| \left(k^{-1}A\right)^t \mathbb{1}_{x_0} - u \right\|_{\mathrm{TV}} \geq 1 - \delta.$$

2. *If $\mathcal{F}$ is a density-Ramanujan expander family, then there exists $n_0$ depending on $\mathcal{F}, \varepsilon_0, \varepsilon_1, \delta$ such that for any $X \in \mathcal{F}$ with order $n > n_0$, all but $\varepsilon_0 n$ of the vertices $x_0 \in X$, and every $t > (1+\varepsilon_1)\frac{k}{k-2}\log_{k-1}(n)$, it holds that*

$$\left\| \left(k^{-1}A\right)^t \mathbb{1}_{x_0} - u \right\|_{\mathrm{TV}} \leq \delta.$$

   *If $X$ is vertex-transitive, then this result holds for all $x \in X$.*

*Remark* 1.20. Notice that the result is weaker than what may be obtained for Ramanujan graphs. For one, the cutoff window $\varepsilon_1 \frac{k}{k-2}\log_{k-1}(n)$ is less sharp than the cutoff window of $O(\sqrt{\log(n)})$ for Ramanujan graphs. In addition, we allow an exceptional set of starting points for which the walk need not exhibit cutoff, making this notion of cutoff weaker than that of Definition 1.1.

We also note that, with high probability, a family of random regular graphs is density-Ramanujan. This recovers the result of Theorem 1.9 with a weaker cutoff window.

The above results generalize to walks on higher-rank complexes as well, using the language and tools of representation theory. This is the content of Chapter 2, where we also state the main theorems of this dissertation.

# Chapter 2

# The Higher-Rank Setup

In this chapter we set up the higher-rank framework in which the main theorems of this dissertation are proved. We begin by a general discussion of Ramanujan and density hypotheses using representation theory (§2.1). In (§2.2), we discuss buildings and their quotients, as well as walks on these complexes. Finally, in §2.3 we survey existing higher-rank cutoff results and state the main theorems of this dissertation.

## 2.1 The Ramanujan Hypothesis and the Density Hypothesis

Let $F$ be a local field, $\mathbf{G}/F$ a semisimple group of $F$-rank $d$, and set $G = \mathbf{G}(F)$. Fix some maximal compact subgroup $K$ of $G$, which we take to be "good" in the sense of Bruhat and Tits when $F$ is non-archimedean. For a representation $\pi \in \widehat{G}$, we consider the matrix coefficient $c_v(g) = \langle \pi(g)v, v \rangle$ for some $K$-finite vector $v$. We use this to define the *rate of decay* of $\pi$ as

$$p(\pi) = \inf\{p \in [2, \infty] \mid c_v \in L^p(G)\}.$$

This quantity does not depend on the choice of $v$. We say a representation $\pi$ is $p$-tempered if $p(\pi) \leq p$. We also refer to 2-tempered representations as tempered, in line with the literature. Let

$\Gamma \leq G$ be a cocompact lattice. We consider the decomposition of $L^2(\Gamma \setminus G)$ as a $G$-representation:

$$L^2(\Gamma \setminus G) = \bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma)\pi.$$

**Definition 2.1.** We say that $\Gamma$ satisfies the *Ramanujan hypothesis* if all infinite-dimensional representations appearing in $L^2(\Gamma \setminus G)$ are tempered.

*Remark* 2.2. Selberg's eigenvalue conjecture is equivalent to the claim that congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ satisfy the Ramanujan hypothesis, once we modify it to only involve representations that appear discretely in $L^2(\Gamma \setminus G)$. It is known that congruence arithmetic lattices in $\mathrm{PGL}_2(\mathbb{Q}_p)$ that arise from a quaternion division algebra over a number field satisfy the Ramanujan hypothesis. For congruence arithmetic lattices in higher-rank groups, the Ramanujan hypothesis often fails, and is not known in many cases where it is expected to hold; see [Li20] or [Sar05] for a survey.

**Definition 2.3.** Let $\{\Gamma_N\}$ be a family of finite-index subgroups of $\Gamma = \Gamma_1$ with $[\Gamma : \Gamma_N] \to \infty$. We say this family satisfies the *density hypothesis* if for every $\varepsilon > 0$, any $p \geq 2$, and any precompact subset $A \subset \widehat{G}$, we have

$$\sum_{\pi \in A, p(\pi) \geq p} m(\pi, \Gamma_N) \ll_{A,\varepsilon} \mathrm{Vol}(\Gamma_N \setminus G)^{2/p+\varepsilon}.$$

We take note of certain variants and special cases of the density hypothesis, as well as references to existing literature on this topic.

- The family $\{\Gamma_N\}$ is said to satisfy the *pointwise* density hypothesis if it satisfies the density hypothesis for each singleton $A = \{\pi\}$. This was the original statement of the density hypothesis by Sarnak and Xue in [SX91], which they proved for $G = \mathrm{SL}_2(\mathbb{R})$ and $G = \mathrm{SL}_2(\mathbb{C})$. Their work was extended by Huntley and Katznelson in [HK93] to non-cocompact congruence arithmetic lattices, and by [FHMM24] to horizontal families of lattices in $\mathrm{SL}_2(\mathbb{R})^a \times \mathrm{SL}_2(\mathbb{C})^b$.
- The family $\{\Gamma_N\}$ is said to satisfy the *cohomological* density hypothesis if it satisfies the density hypothesis for $A = \{\pi\}$ for cohomological representations $\pi$. See [CM13], [Mar14], [Mar16], [MS19], and [EGGG23] for work in this vein, and applications to growths of Betti

numbers.

- If $G$ is a $p$-adic group or a rank 1 Lie group, then the set $\widehat{G}_K$ of $K$-spherical representations is precompact, and the family $\{\Gamma_N\}$ is said to satisfy the *spherical* density hypothesis if it satisfies the density hypothesis for $A = \widehat{G}_K$. For $p$-adic groups, the spherical density hypothesis implies the general density hypothesis. For higher rank Lie groups, $\widehat{G}_K$ is not necessarily precompact, and so a modification of the spherical density hypothesis is needed, which is not implied by the general density hypothesis. See [GK23] for a more detailed discussion.
- The Satake variant of the pointwise density hypothesis replaces the exponent $\frac{2}{p(\pi)}$ with $1 - \frac{\sigma_\pi}{\sigma_1}$, where $\sigma_\pi$ is the maximum real part of the Satake parameters of $\pi$. See [Man22], [Blo23], [Ass23], [AB24], and [JK24] for results in this vein.

*Remark* 2.4. We consider the extreme cases of $p = 2$ and $p = \infty$, in the setting of the density hypothesis.

1. If $\pi$ is tempered, the pointwise density hypothesis states $m(\pi, \Gamma_N) \ll_\varepsilon \mathrm{Vol}(\Gamma_N \setminus G)^{1+\varepsilon}$. This is $\varepsilon$-worse than the trivial bound $m(\pi, \Gamma_N) \ll \mathrm{Vol}(\Gamma_N \setminus G)$, which is sharp for $\pi$ a discrete series.
2. If $\pi$ is the trivial representation, then $m(\pi, \Gamma_N) = 1$. Since $p(\pi) = \infty$, the pointwise density hypothesis says $m(\pi, \Gamma_N) \ll \mathrm{Vol}(\Gamma_N \setminus G)^{\varepsilon}$, so this is again $\varepsilon$-worse than what is known.

The pointwise density hypothesis interpolates between the two cases above, and provides a non-trivial bound for $m(\pi, \Gamma_N)$ when $\pi$ is neither trivial nor tempered.

Sarnak conjectures that the density hypothesis should hold for families of congruence arithmetic lattices even when the Ramanujan hypothesis does not (see [GK23] for a detailed discussion). This conjecture is more tractable than the higher-rank Ramanujan-Petersson conjecture, and can substitute for it in many applications.

One such application is the cutoff phenomenon, which we have already seen for graphs via Theorem 1.19. To see how the Ramanujan and density hypotheses for higher rank groups apply to the study of cutoff, we now discuss Bruhat-Tits buildings.

## 2.2 Bruhat-Tits Buildings

Assume now that $F$ is non-archimedean with ring of integers $\mathcal{O}$, uniformizer $\varpi$, and let $q$ be the order of its residue field $\mathcal{O}/(\varpi)$. We may construct from $G$ an infinite, contractible, $d$-dimensional simplicial complex upon which it acts, known as the *Bruhat-Tits building* $\mathcal{B}$. For any $0 \leq i \leq d$, we write $\mathcal{B}^i$ for the $i$-skeleton of $\mathcal{B}$. We also refer to $d$-dimensional cells as *chambers*. Fix some chamber $\sigma_0$ and let $I$ be its pointwise stabilizer, known as an *Iwahori subgroup* of $G$. As $G$ acts transitively on chambers, $I$ is unique up to conjugation. We may identify the ordered $d$-skeleton of $\mathcal{B}$ with $G/I$.

We may define the analog of being a Ramanujan graph or expander graph for finite quotients of $\mathcal{B}$, using the notion of $p$-temperedness.

**Definition 2.5.** Let $\Gamma$ be a torsion-free, cocompact subgroup of $G$. The finite complex $X = \Gamma \setminus \mathcal{B}$ is an $L^p$-expander complex if every $I$-spherical $\pi \in \widehat{G}$ appearing with non-zero multiplicity in $L^2(\Gamma \setminus G)$ is either finite-dimensional or $p$-tempered. When $p = 2$, we say $X$ is a *Ramanujan complex*. A family $\{X_N\}$ of such complexes is said to be an *expander family* if there exists some $p$ (independent of $N$) such that each $X_N$ is an $L^p$-expander.

*Remark* 2.6. The reader is encouraged to consult [K16] for an in-depth discussion of these notions. For $\mathbf{G} = \mathrm{PGL}_2$, $\mathcal{B}$ is the $(q+1)$-regular tree, in which case the preceding definitions of expansion and Ramanujan complexes agree with the familiar definitions from spectral theory. If $\mathbf{G}$ is almost $F$-simple with $F$-rank at least 2, the work of Oh [Oh02] on quantitative property (T) ensures that all complexes $\Gamma \setminus \mathcal{B}$ are $L^p$-expanders for some $p$ depending only on $\mathbf{G}$.

**Definition 2.7.** The family of complexes $X_N = \Gamma_N \setminus \mathcal{B}$ is said to be *density-Ramanujan* if the family of lattices $\Gamma_N$ satisfies the density hypothesis with $A$ restricted to be a subset of the $I$-spherical representations.

We now move to understand random walks on quotients of buildings. Let $\mathfrak{C}$ be a subset of the cells of $\mathcal{B}$ and let $T : \mathfrak{C} \to 2^{\mathfrak{C}}$ be a branching operator. This allows us to form a digraph $\mathcal{D}_{T,\mathfrak{C}}$ with vertex set $\mathfrak{C}$, and for which $x \to y$ is a directed edge whenever $y \in T(x)$. If $\mathcal{D}_{T,\mathfrak{C}}$ is a $k$-in-regular and $k$-out-regular digraph, we say that $T$ is a *$k$-regular branching operator*, which will be the main

case in which we are interested. We say $T$ is collision-free if for any $x, y \in \mathcal{D}_{T,\mathfrak{C}}$, there is at most 1 directed path from $x$ to $y$.

The adjacency operator of $\mathcal{D}_{T,\mathfrak{C}}$, acting on $L^2(\mathfrak{C})$, is given by

$$(A_{\mathcal{D}_{T,\mathfrak{C}}}\varphi)(x) = \sum_{y \in T(x)} \varphi(y).$$

If $\mathfrak{C}$ is $G$-invariant and $T$ is $G$-equivariant[1], then $T$ induces a branching operator on quotients $\mathfrak{X} = \Gamma \setminus \mathfrak{C}$. We denote the associated digraphs by $\mathcal{D}_{T,\mathfrak{X}}$, and their adjacency operators acting on $L^2(\mathfrak{X})$ take a similar form:

$$(A_{\mathcal{D}_{T,\mathfrak{X}}} f)(x) = \sum_{y \in T(x)} f(y).$$

When there is no ambiguity, we may identify $T$ with the above-described adjacency operators and write $T\varphi$ and $Tf$. We denote by $\widetilde{T}$ the stochastic operator associated to $T$, which equals $k^{-1}T$ if $T$ is $k$-regular.

## 2.3 Cutoff in Higher Rank

Having laid the groundwork, we now state a collection of results establishing cutoff in higher rank, culminating in the main theorems of this dissertation. Throughout, unless otherwise specified, we take $F$ to be a non-archimedean local field, $\mathbf{G}/F$ a semisimple group, $G = \mathbf{G}(F)$, and $\mathcal{B} = \mathcal{B}(G)$.

Theorem 1.14 was generalized to walks on higher-dimensional complexes by Lubetzky, Lubotzky, and Parzanchevski in [LLP20]:

**Theorem 2.8.** *Assume $\boldsymbol{G}$ is simple and let $T$ be a $k$-regular, collision-free, and $G$-equivariant branching operator on a subset of cells $\mathfrak{C} \subset \mathcal{B}$. Let $X_N = \Gamma_N \setminus \mathcal{B}$ be a family of Ramanujan complexes, $\mathfrak{X}_N = \Gamma_N \setminus \mathfrak{C} \subset X_N$, and assume that $\mathcal{D}_{T,\mathfrak{X}_N}$ is strongly connected. Then there exists some constant $M_G$ such that the walk associated to $T$ on $\mathfrak{X}_N$, modified in its first step to perform $U_0 = \mathrm{Uniform}(\{1, \dots, M_G\})$ steps, exhibits cutoff at time $\log_k(|\mathfrak{X}_N|)$ and with window $O_G(\log_k \log(|\mathfrak{X}_N|))$.*

[1] Sometimes called "geometric".

*Remark* 2.9. The modification of the walk accounts for the possibility that it is periodic.

The idea of the argument is to show that there exists some $r = r(G)$ such that $\mathcal{D}_{T,\mathfrak{X}}$ is an $r$-normal Ramanujan digraph when $\Gamma \setminus \mathcal{B}$ is a Ramanujan complex. The result follows from the fact random walk on a family of $r$-normal Ramanujan digraphs exhibits cutoff.

For $\mathbf{G} = \mathrm{PGL}_d$, we have *geodesic flow* operators in each dimension, which satisfy the hypotheses of Theorem 2.8 and therefore exhibit cutoff. This generalizes NBRW on edges, which is the 1-dimensional geodesic flow on $\mathcal{B}(\mathrm{PGL}_2)$ but which is not itself collision-free in higher rank.

The case of SRW was established by Chapman and Parzanchevski in [CP22].

**Theorem 2.10.** *Set $\mathbf{G} = \mathrm{PGL}_d$. Then there exists $C_{d,q}$ such that the family of Ramanujan quotients $X_N$ exhibits cutoff with respect to SRW on their 0-vertices $X_N^0$ at time $C_{d,q} \log_q(|X_N^0|)$ and with window $O(\sqrt{\log(|X_N^0|)})$.*

This uses a combination of the geodesic flow operators in each dimension in place of the NBRW random walk on edges for the reduction of Theorem 1.13 to Theorem 1.14.

The following result of Evra, Gerbelli-Gauthier, and Gustafsson [EGGG23] exhibits cutoff for a family of walks in higher rank assuming only the density hypothesis.

**Theorem 2.11.** *Let $\Gamma$ be an arithmetic lattice in $G$ and let $\{\Gamma_N\}$ be the family of principal congruence subgroups of $\Gamma$. Assume the family of complexes $X_N = \Gamma_N \setminus \mathcal{B}$ are expanders and density-Ramanujan. Then for any $\varepsilon_1 > 0$, the quotients $X_N = \Gamma_N \setminus \mathcal{B}$ exhibit cutoff with respect to NBRW on chambers at time $\log_k(|X_N^d|)$ and with window $\varepsilon_1 \log_k(|X_N^d|)$.*

*Remark* 2.12. We expect that the $X_N$ are automatically an expander family due to the $\Gamma_N$ being congruence arithmetic, but we have not checked this carefully.

As with [LLP20], the argument proceeds by using analysis of matrix coefficients to obtain a spectral bound for $T$. Due to the presence of non-tempered representations in $L^2(\Gamma_N \setminus G)$, the (non-trivial) spectral radius of $T$ acting on $L^2(X_N^d)$ is larger than $\sqrt{k}$, leading to a worse bound for $\|T^t\|$. The density hypothesis compensates by controlling the norm of the indicator function $\mathbb{1}_x$ projected onto

the subspaces of $L^2(\mathfrak{X}_N)$ associated to $p$-tempered representations for $p \geq 2$. The assumption that the $\Gamma_N$ are principal congruence arithmetic implies that the $X_N$ have large automorphism group, which allows for cutoff to hold for any starting point[2].

Our first main theorem generalizes Theorem 2.11 to collision-free walks in any dimension, and allows for the $\Gamma_N$ to be any family of finite-index subgroups of some fixed torsion-free cocompact lattice $\Gamma$. The argument proceeds along a similar outline, using the framework of [LLP20] to allow treatment of more general $T$ (particularly to accommodate periodicity), and with modifications to allow for the lack of a "near-transitivity" assumption.

**Theorem 2.13.** *Assume* $\mathbf{G}$ *is semisimple. Let* $T$ *be a* $k$*-regular, collision-free, and* $G$*-equivariant branching operator on* $\mathfrak{C} \subset \mathcal{B}$*. Let* $X_N = \Gamma_N \backslash \mathcal{B}$ *be a density-Ramanujan expander family, write* $\mathfrak{X}_N = \Gamma_N \backslash \mathfrak{C} \subset X_N$*, and assume that* $\mathcal{D}_{T,\mathfrak{X}_N}$ *is strongly connected. Then there exists some constant* $M_G$ *such that for any* $\varepsilon_0, \varepsilon_1 > 0$*, the walk induced by* $T$ *on the* $\mathfrak{X}_N$*, modified in its first step to perform* $U_0 = \mathit{Uniform}(\{1, \dots, M_G\})$ *steps, exhibits cutoff at time* $\log_k(|\mathfrak{X}_N|)$ *with window* $\varepsilon_1 \log_k(|\mathfrak{X}_N|)$ *at all but possibly* $\varepsilon_0|\mathfrak{X}_N|$ *starting points.*

Our second main theorem handles the case of SRW, replicating the argument of [CP22] in this setting.

**Theorem 2.14.** *Set* $\mathbf{G} = \mathrm{PGL}_d$ *and let* $X_N = \Gamma_N \backslash \mathcal{B}$ *be a density-Ramanujan expander family. Then there exist constants* $C_{d,q}$ *such that for any* $\varepsilon_0, \varepsilon_1 > 0$*, simple random walk on the* $X_N$*, modified in its first step to be lazy if the* $X_N$ *are bipartite graphs, exhibits the cutoff phenomenon at time* $C_{d,q} \log_q(|X_N^0|)$ *with window* $\varepsilon_1 \log(|X_N^0|)$ *for all but possibly* $\varepsilon_0|X_N^0|$ *starting points.*

Remark 1.20 holds here as well. The proof of Theorem 2.13 occupies Chapter 3, and the proof of Theorem 2.14 occupies Chapter 4.

[2] Cf. Theorem 1.19, in which the walk may fail to exhibit cutoff at an exceptional set of starting points in the absence of the assumption that the graphs are vertex-transitive

# Chapter 3

# Cutoff for Collision-Free Walks

The aim of this chapter is to prove Theorem 2.13. For convenience, we recall the setup: we fix a set of simplices $\mathfrak{C} \subset \mathcal{B}$ and a $k$-regular, collision-free, and $G$-equivariant branching operator $T : \mathfrak{C} \to \binom{\mathfrak{C}}{k}$. We also fix a lattice $\Gamma = \Gamma_1$ and consider a family of finite-index subgroups $\Gamma_N$ with $[\Gamma : \Gamma_N] \to \infty$, which will be assumed to satisfy the density hypothesis. We set $\mathfrak{X}_N = \Gamma_N \setminus \mathfrak{C}$, and write $n = |\mathfrak{X}_N|$. We fix a maximal chamber $\sigma_0$ and let $I$ be the associated Iwahori subgroup.

## 3.1 Analysis of $T$

Fix a collection $\Sigma$ of faces of $\sigma_0$ such that $\mathfrak{C} = \bigsqcup_{\sigma \in \Sigma} G\sigma$. Then every cell of $\mathfrak{X}_N$ is of the form $\Gamma_N g\sigma$ for some $g \in G$ and $\sigma \in \Sigma$. For every $\sigma, \tau \in \Sigma$, we pick a set $S^{\sigma,\tau} \subset G$ such that

$$T(\sigma) \cap G\tau = S^{\sigma,\tau}\tau.$$

The elements in $S^{\sigma,\tau}$ are to be chosen "without repetition": each element in $T(\sigma) \cap G\tau$ should be written uniquely as $s\tau$ for $s \in S^{\sigma,\tau}$. In this notation, the adjacency operator acts on $L^2(\mathfrak{C})$ by

$$(T\varphi)(g\sigma) = \sum_{\tau \in \Sigma} \sum_{s \in S^{\sigma,\tau}} \varphi(gs\tau),$$

and on $L^2(\mathfrak{X}_N)$ as

$$(Tf)(\Gamma_N g\sigma) = \sum_{\tau\in\Sigma}\sum_{s\in S^{\sigma,\tau}} f(\Gamma_N gs\tau).$$

Identifying $f \in L^2(\mathfrak{X}_N)$ with a collection $f_\sigma \in L^2(\Gamma_N \setminus G\sigma)$ for $\sigma \in \Sigma$, we have

$$L^2(\mathfrak{X}_N) \cong L^2(\Gamma_N \setminus G)^\Sigma.$$

Here $V^\Sigma$ denotes, for any $G$-representation $V$, the space

$$\bigoplus_{\sigma\in\Sigma} V^{G_\sigma} \leq \bigoplus_{\sigma\in\Sigma} V.$$

The latter space is a representation of the matrix algebra $M_{\Sigma\times\Sigma}(\mathbb{C}G)$. Under this identification, $T$ acts on $L^2(\Gamma_N \setminus G)^\Sigma$ by the matrix $\mathcal{A} \in M_{\Sigma\times\Sigma}(\mathbb{C}G)$ where

$$(\mathcal{A})_{\sigma,\tau} = \sum_{s\in S^{\sigma,\tau}} s.$$

In particular, $L^2(\Gamma_N \setminus G)^\Sigma$ is $\mathcal{A}$-invariant. In addition, the decomposition of $L^2(\Gamma_N \setminus G)$ into irreducible representations descends as

$$L^2(\mathfrak{X}_N) \cong \bigoplus_\pi m(\pi, \Gamma_N)\pi^\Sigma,$$

and this decomposition is respected by $T$. We now wish to understand the spectrum of $T$ on each component $\pi^\Sigma$ in terms of $p(\pi)$, for which we need the following general auxiliary lemma about collision-free operators. This is a direct generalization of the $p = 2$ case, carried out in [LLP20, Proposition 2.3].

**Lemma 3.1.** *Let $X$ be a countable set and let $T : X \to \binom{X}{k}$ be a collision-free operator, which we identify with the associated adjacency operator on functions $X \to \mathbb{C}$. Then for any $p \geq 1$ and any $\lambda$ in the $p^+$ point spectrum of $T$, we have $|\lambda| \leq k^{1-1/p}$.*

*Remark* 3.2. Recall that the $p^+$ point spectrum of $T$, denoted $\mathrm{Pspec}_{p^+}(T)$, is the set of $\lambda \in \mathbb{C}$ for which $Tf = \lambda f$ for some $f$ contained in $L^{p+\varepsilon}$ for every $\varepsilon > 0$.

*Proof.* Assume $\lambda \in \mathrm{Pspec}_{p^+}(T)$ and pick $f$ as in Remark 3.2. Pick some $v \in X$ for which $f(v) \neq 0$, and scale $f$ so that $f(v) = 1$. Then for any $\varepsilon > 0$ and any non-negative integer $j$, we have

$$\sum_{w \in T^j(v)} |f(w)|^{p+\varepsilon} \geq \left| \frac{\lambda^{p+\varepsilon}}{k^{p-1+\varepsilon}} \right|^j.$$

For $j = 0$, this is clear. Since $T$ is collision-free, $T(w)$ and $T(w')$ are disjoint for any distinct $w, w' \in T^j(v)$, so that

$$\sum_{w \in T^{j+1}(v)} |f(w)|^{p+\varepsilon} = \sum_{u \in T^j(v)} \sum_{w \in T(u)} |f(w)|^{p+\varepsilon}.$$

The Hölder conjugate of $p + \varepsilon$ is $\frac{p+\varepsilon}{p-1+\varepsilon}$, so applying Hölder's inequality and the triangle inequality, we see by induction that:

$$\begin{aligned}
\sum_{u \in T^j(v)} \sum_{w \in T(u)} |f(w)|^{p+\varepsilon} &\geq \frac{1}{k^{p-1+\varepsilon}} \sum_{u \in T^j(v)} \left( \sum_{w \in T(u)} |f(w)| \right)^{p+\varepsilon} \\
&\geq \frac{1}{k^{p-1+\varepsilon}} \sum_{u \in T^j(v)} |Tf(u)|^{p+\varepsilon} \\
&= \frac{1}{k^{p-1+\varepsilon}} \sum_{u \in T^j(v)} |\lambda f(u)|^{p+\varepsilon} \\
&= \frac{|\lambda|^{p+\varepsilon}}{k^{p-1+\varepsilon}} \sum_{u \in T^j(v)} |f(u)|^{p+\varepsilon} \\
&\geq \left| \frac{\lambda^{p+\varepsilon}}{k^{p-1+\varepsilon}} \right|^{j+1}.
\end{aligned}$$

We again use that $T$ is collision-free to see that the $T^j(v)$ are disjoint, so:

$$\|f\|_{p+\varepsilon}^{p+\varepsilon} \geq \sum_{j=0}^{\infty} \sum_{w \in T^j(v)} |f(w)|^{p+\varepsilon} \geq \sum_{j=0}^{\infty} \left| \frac{\lambda^{p+\varepsilon}}{k^{p-1+\varepsilon}} \right|^j.$$

This geometric series is finite, so $|\lambda| < k^{\frac{p-1+\varepsilon}{p+\varepsilon}}$, so $|\lambda| \leq k^{1-1/p}$. □

This allows us to extend the argument of [LLP20, Proposition 4.1] and obtain a spectral bound for $T$ restricted to $\pi^\Sigma$ when $\pi$ is non-tempered.

**Proposition 3.3.** *Every eigenvalue $\lambda$ of $T|_{\pi^\Sigma}$ satisfies $|\lambda| \leq k^{1-1/p(\pi)}$.*

*Proof.* Let $f = (f_\sigma)_{\sigma\in\Sigma}$ be in the $\lambda$-eigenspace of $T$, and fix some $\xi \in \Sigma$ such that $f_\xi \neq 0$. Then for each $\sigma \in \Sigma$, define the matrix coefficient $\varphi_\sigma : G \to \mathbb{C}$ by

$$\varphi_\sigma(g) = \langle g f_\sigma, f_\xi\rangle = \int_{\Gamma_N\backslash G} f_\sigma(\Gamma_N x g)\overline{f_\xi(\Gamma_N x)}d(\Gamma_N x).$$

Since $f_\sigma$ is right $G_\sigma$-invariant, so is $\varphi_\sigma$, so it defines a function on $G\sigma$. The collection of $\varphi_\sigma$ patch to a function $\varphi : \mathfrak{C} \to \mathbb{C}$. Then $\varphi \in L^{p(\pi)+\varepsilon}(\mathfrak{C})$ for every $\varepsilon > 0$ since each $\varphi_\sigma$ is, and is non-zero since $\varphi(\xi) = \varphi_\xi(e) = \|f_\xi\|^2 \neq 0$. In fact, we have

$$\begin{aligned}(T\varphi)(g\sigma) &= \sum_{\tau\in\Sigma}\sum_{s\in S^{\sigma,\tau}} \varphi(gs\tau) = \sum_{\tau\in\Sigma}\sum_{s\in S^{\sigma,\tau}} \varphi_\tau(gs) = \sum_{\tau\in\Sigma}\sum_{s\in S^{\sigma,\tau}} \langle gsf_\tau, f_\xi\rangle \\ &= \left\langle \sum_{\tau\in\Sigma} \mathcal{A}_{\sigma,\tau} f_\tau, g^{-1} f_\xi \right\rangle = \langle \lambda f_\sigma, g^{-1} f_\xi\rangle = \lambda\varphi_\sigma(g) = \lambda\varphi(g\sigma),\end{aligned}$$

so $\varphi$ is an eigenfunction of $T$ acting on $L^{p(\pi)+\varepsilon}(\mathfrak{C})$ for each $\varepsilon$, meaning $\lambda \in \mathrm{Pspec}_{p(\pi)^+}(T)$. By Lemma 3.1, this implies that $|\lambda| \leq k^{1-1/p(\pi)}$. □

If the $\mathfrak{X}_N$ form an expander family, then the only representations $\pi$ satisfying $p(\pi) = \infty$ for which $\pi^\Sigma$ appears in the above decomposition for $L^2(\mathfrak{X}_N)$ are the one-dimensional characters. We refer to the eigenvalues of $T$ associated to such representations as the *trivial eigenvalues*.

**Proposition 3.4.** *Let $M$ be the period of the matrix of $T$ acting on $L^2(\mathfrak{X}_N)$. Then the trivial eigenvalues of $T$ are simple, and are of the form $\zeta k$ as $\zeta$ ranges over the $M$th roots of unity.*

*Proof.* Note that the matrix of $T$ is irreducible (and thus has a finite period) by the assumption that the digraphs $\mathcal{D}_{T,\mathfrak{X}_N}$ are strongly connected. By [LLP20, Proposition 4.3], the trivial eigenvalues $\lambda$ of $T$ satisfy $|\lambda| = k$. The result then follows from the Perron-Frobenius theorem for irreducible matrices. □

*Remark* 3.5. While Lubetzky, Lubotzky, and Parzanchevski restrict attention to Ramanujan complexes, their analysis of the trivial eigenvalues does not rely on this assumption.

As this bound on the spectral radius of $T|_{\pi^\Sigma}$ depends only on $p(\pi)$, we are led to define:

$$V(p,N) = \bigoplus_{p(\pi)=p} m(\pi, \Gamma_N)\pi^\Sigma,$$

so that

$$L^2(\mathfrak{X}_N) \cong \bigoplus_{p\in[2,\infty]} V(p,N).$$

**Theorem 3.6.** *There exists $r = r(G)$ such that for any $p \in [2,\infty]$, we have*

$$\|T^t|_{V(p,N)}\| \ll t^r k^{t(1-1/p)},$$

*for all $N$ and $t$.*

*Proof.* We first note that the operator norm of $T^t$ restricted to $V(p,N)$ is the supremum of the operator norms of its restrictions to the components $\pi^\Sigma$. Thus, it suffices to show that

$$\|T^t|_{\pi^\Sigma}\| \ll t^r k^{t(1-1/p)},$$

whenever $p(\pi) = p$. Following the proof of [Par20, Proposition 4.1], we see that this follows from the eigenvalue bound above, as well as the fact $\dim(\pi^\Sigma)$ is bounded by some $r = r(G)$, which is proven in [LLP20, Proposition 4.5]. □

## 3.2 Projecting to $V(p,N)$

Unlike in the Ramanujan case, we do not have an adequate global bound on $\|T^t\|$. Here we use the density hypothesis to bound the norm of the projection of the indicator function $\mathbb{1}_x$ to the subspaces $V(p,N)$. We denote this projection operator by $\mathcal{P}_{p,N}$.

**Lemma 3.7.** *For any $p \in [2,\infty)$ and any $\varepsilon > 0$, we have:*

$$\sum_{p'\in[p,\infty)} \sum_{y\in\mathfrak{X}_N} \|\mathcal{P}_{p',N}(\mathbb{1}_y)\|_2^2 \ll_\varepsilon n^{2/p+\varepsilon}.$$

*Proof.* Write $M$ for the matrix of $\mathcal{P}_{p',N}$ in the basis $\{\mathbb{1}_y \mid y \in \mathfrak{X}_N\}$. Then $M$ is self-adjoint and idempotent, so

$$M_{yy} = \langle M\mathbb{1}_y, \mathbb{1}_y\rangle = \langle M^2\mathbb{1}_y, \mathbb{1}_y\rangle = \langle M^*M\mathbb{1}_y, \mathbb{1}_y\rangle = \langle M\mathbb{1}_y, M\mathbb{1}_y\rangle = \|\mathcal{P}_{p',N}(\mathbb{1}_y)\|_2^2.$$

Thus, we have

$$\sum_{y\in\mathfrak{X}_N} \|\mathcal{P}_{p',N}(\mathbb{1}_y)\|_2^2 = \mathrm{Tr}(M) = \dim(V(p',N)).$$

We apply the density hypothesis to get

$$\sum_{p'\in[p,\infty)} \sum_{y\in\mathfrak{X}_N} \|\mathcal{P}_{p',N}(\mathbb{1}_y)\|_2^2 \ll_\varepsilon n^{2/p+\varepsilon}.$$

□

*Remark* 3.8. One may hope to conclude that the inequality

$$\sum_{p'\in[p,\infty)} \|\mathcal{P}_{p',N}(\mathbb{1}_y)\|_2^2 \ll_\varepsilon n^{-1+2/p+\varepsilon},$$

holds for each $y \in \mathfrak{X}_N$, since it holds "on average". Assuming an appropriate transitivity-type hypothesis, this is true, and allows for cutoff to hold at any starting point. The reader is encouraged to consult Lemma 8.21 and Proposition 8.22 of [EGGG23], from which the proof of the preceding lemma is drawn. In their setting, Proposition 8.20 furnishes the transitivity.

## 3.3 Cutoff

Write

$$P_M(\widetilde{T}) = \frac{1}{M}\sum_{j=1}^{M} \widetilde{T}^j,$$

where $M$ is to be determined. The time $t+1$ distribution $\mu_{N,x}^{t+1}$ of the walk starting at $x$, modified to perform Uniform$(\{1,\dots,M\})$ steps in its first step, is given by

$$\widetilde{T}^t P_M(\widetilde{T})\mathbb{1}_x.$$

The distance to stationary of the time $t+1$ distribution of the walk starting at $x$ is bounded by

$$d_x(t+1,N) \coloneqq \|\mu_{N,x}^{t+1} - u\|_1 \leq \|\mathcal{P}_{f,N}(\mu_{N,x}^{t+1})\|_1 + \|\mathcal{P}_{\infty,N}(\mu_{N,x}^{t+1}) - u\|_1.$$

where $\mathcal{P}_{f,N}$ is the projection to $\bigoplus_{p<\infty} V(p,N)$.

**Proposition 3.9.** *For any $\delta > 0$, there exists some $t_\infty$ such that for all $N$, all $t \geq t_\infty$, and all $x \in \mathfrak{X}_N$, we have*

$$\|\mathcal{P}_{\infty,N}(\mu_{N,x}^{t+1}) - u\|_1 \leq \delta/2.$$

*Proof.* We have

$$V(\infty,N) = \bigoplus_{\pi \in \mathrm{Hom}(G,\mathbb{C}^\times)} m(\pi,\Gamma_N)\pi^\Sigma \cong L^2(\Gamma_N G' \setminus \mathfrak{C}),$$

where $G'$ is the derived subgroup of $G$. The action of $\widetilde{T}$ on this space is irreducible, as it is the quotient of its action on $\mathfrak{X}_N$ under the natural map $\mathfrak{X}_N \to \Gamma_N G' \setminus \mathfrak{C}$. In addition, there are only finitely many possibilities for the subgroup $\Gamma_N G'$ of $G$, since it contains the finite-index subgroup $G'$. Thus, there are only finitely many possibilities for the space $\Gamma_N G' \setminus \mathfrak{C}$, so we choose $M$ such that the modified walk is aperiodic on all of them, and see that there exists some $t_\infty$ (independent of $N$) such that all the walks are $\delta/2$-mixed whenever $t \geq t_\infty$. □

We now move to control the term $\|\mathcal{P}_{f,N}(\mu_{N,x}^{t+1})\|_1$, averaged over $x \in \mathfrak{X}_N$.

**Proposition 3.10.** *Fix $r$ as in Theorem 3.6. There exists a constant $C \geq 2$, independent of $N$, such that for any $\varepsilon > 0$ and any $N$, we have the following bounds:*

- *If $t = \log_k(n)$, then*

$$\frac{1}{n}\sum_{x \in \mathfrak{X}_N} \|\mathcal{P}_{f,N}(\mu_{N,x}^{t+1})\|_1^2 \ll_\varepsilon n^\varepsilon t^{2r+1}.$$

- *If* $t < \log_k(n)$, *then*

$$\frac{1}{n}\sum_{x\in\mathfrak{X}_N}\|\mathcal{P}_{f,N}(\mu_{N,x}^{t+1})\|_1^2 \ll_\varepsilon t^{2r}\left(\frac{n}{k^t}+n^\varepsilon t\log(k)\frac{n/k^t}{\log(n/k^t)}\right).$$

- *If* $t > \log_k(n)$, *then*

$$\frac{1}{n}\sum_{x\in\mathfrak{X}_N}\|\mathcal{P}_{f,N}(\mu_{N,x}^{t+1})\|_1^2 \ll_\varepsilon t^{2r}\left(\frac{n}{k^t}+n^\varepsilon t\log(k)\frac{(n/k^t)^{2/C}}{\log(k^t/n)}\right).$$

*Proof.* For each $x \in \mathfrak{X}_N$, we have

$$\begin{aligned}\|\mathcal{P}_{f,N}(\mu_{N,x}^{t+1})\|_1^2 &\le n\|\mathcal{P}_{f,N}(\mu_{N,x}^{t+1})\|_2^2\\ &\le n\sum_{p<\infty}\|\widetilde{T}^t P_M(\widetilde{T})|_{V(p,N)}\|^2\ \|\mathcal{P}_{p,N}(\mathbb{1}_x)\|_2^2.\end{aligned}$$

By Theorem 3.6,

$$\|\widetilde{T}^t P_M(\widetilde{T})|_{V(p,N)}\| \le \frac{1}{M}\sum_{j=1}^{M}\|\widetilde{T}^{t+j}|_{V(p,N)}\| \ll \frac{1}{M}\sum_{j=1}^{M}(t+j)^r k^{-(t+j)/p} \ll t^r k^{-t/p}.$$

Averaging over $x \in \mathfrak{X}_N$, we have

$$\begin{aligned}\frac{1}{n}\sum_{x\in\mathfrak{X}_N}\|\mathcal{P}_{f,N}(\mu_{N,x}^{t+1})\|_1^2 &\le \frac{1}{n}\sum_{x\in\mathfrak{X}_N} n\sum_{p<\infty}\|\widetilde{T}^t P_M(\widetilde{T})|_{V(p,N)}\|^2\|\mathcal{P}_{p,N}(\mathbb{1}_x)\|_2^2\\ &= \sum_{p<\infty}\|\widetilde{T}^t P_M(\widetilde{T})|_{V(p,N)}\|^2\sum_{x\in\mathfrak{X}_N}\|\mathcal{P}_{p,N}(\mathbb{1}_x)\|_2^2\\ &\ll t^{2r}\sum_{p<\infty}k^{-2t/p}\sum_{x\in\mathfrak{X}_N}\|\mathcal{P}_{p,N}(\mathbb{1}_x)\|_2^2.\end{aligned}$$

Define the quantities

- $a(p) = \sum_{x\in\mathfrak{X}_N}\|\mathcal{P}_{p,N}(\mathbb{1}_x)\|_2^2$,
- $A(p) = \sum_{p'\in[p,\infty)} a(p')$, and
- $f(p) = k^{-2t/p}$.

Integration by parts gives

$$\sum_{p<\infty} f(p)a(p) = f(2)A(2) + \int_2^\infty A(p)f'(p)dp.$$

We now use the assumption that the $\mathfrak{X}_N$ are an expander family, so there exists some $C$ independent of $N$ such that $V(p,N) = 0$ for all $p > C$. Thus, $A(p) = 0$ for all $p > C$, and we may truncate the integral at $C$ to obtain

$$\sum_p f(p)a(p) = k^{-t}A(2) + \int_2^C \left( \sum_{p' \in [p,\infty)} \sum_{x \in \mathfrak{X}_N} \|\mathcal{P}_{p',N}(\mathbb{1}_x)\|_2^2 \right) \frac{2t\log(k)}{p^2} k^{-2t/p} dp.$$

We bound the sum within the integral by Lemma 3.7, and use that $A(2) \leq n$ to see that

$$\sum_p f(p)a(p) \ll_\varepsilon \frac{n}{k^t} + n^\varepsilon \; 2t\log(k) \int_2^C p^{-2} n^{2/p} k^{-2t/p} dp.$$

If $t = \log_k(n)$, then $n/k^t = 1$ and the integral evaluates to

$$\int_2^C p^{-2} dp = \frac{1}{2} - \frac{1}{C} \leq \frac{1}{2}.$$

Substituting this gives the desired bound:

$$\frac{1}{n} \sum_{x \in \mathfrak{X}_N} \|\mathcal{P}_{f,N}(\mu_{N,x}^{t+1})\|_1^2 \ll_\varepsilon t^{2r}(1 + n^\varepsilon t\log(k)) \ll n^\varepsilon t^{2r+1}.$$

Otherwise, we apply the change of variables $u = 1/p$ and see that the integral evaluates to

$$\frac{n/k^t - (n/k^t)^{2/C}}{2\log(n/k^t)}.$$

If $t < \log_k(n)$, then this is at most

$$\frac{n/k^t}{2\log(n/k^t)}.$$

Substituting this gives the desired bound:

$$\frac{1}{n}\sum_{x\in\mathfrak{X}_N}\|\mathcal{P}_{f,N}(\mu_{N,x}^{t+1})\|_1^2 \ll_\varepsilon t^{2r}\left(\frac{n}{k^t}+n^\varepsilon t\log(k)\frac{n/k^t}{\log(n/k^t)}\right).$$

If $t > \log_k(n)$, then we note that $\log(n/k^t) < 0$ and $n/k^t < (n/k^t)^{2/C}$, so we instead bound the integral by

$$\frac{(n/k^t)^{2/C}}{2\log(k^t/n)}.$$

Substituting this gives the desired bound:

$$\frac{1}{n}\sum_{x\in\mathfrak{X}_N}\|\mathcal{P}_{f,N}(\mu_{N,x}^{t+1})\|_1^2 \ll_\varepsilon t^{2r}\left(\frac{n}{k^t}+n^\varepsilon t\log(k)\frac{(n/k^t)^{2/C}}{\log(k^t/n)}\right).$$

□

**Proposition 3.11.** *Fix $\varepsilon_0, \varepsilon_1, \delta > 0$ and set $t_u = (1+\varepsilon_1)\log_k(n)$. For large enough $N$, all but at most $\varepsilon_0 n$ starting points $x \in \mathfrak{X}_N$ satisfy $\|\mathcal{P}_{f,N}(\mu_{N,x}^{t_u+1})\|_1 \le \delta/2$.*

*Proof.* At time $t_u = (1+\varepsilon_1)\log_k(n)$, we have $k^{-t_u} \le n^{-(1+\varepsilon_1)}$, so that

$$\frac{1}{n}\sum_{x\in\mathfrak{X}_N}\|\mathcal{P}_{f,N}(\mu_{N,x}^{t_u+1})\|_1^2 \ll_\varepsilon t_u^{2r}\left(n^{-\varepsilon_1}+\frac{t_u\log(k)}{\varepsilon_1\log(n)}n^{\varepsilon-2\varepsilon_1/C}\right).$$

We denote the right hand expression by $B_{\varepsilon,\varepsilon_1}(n)$ and note that it exhibits polynomial decay in $n$ if $\varepsilon < 2\varepsilon_1/C$. Treating $\mathfrak{X}_N$ as a probability space with the uniform distribution, the left hand expression may be seen as the expectation of the random variable $\|\mathcal{P}_{f,N}(\mu_{N,x}^{t_u+1})\|_1^2$. By Markov's inequality, we have

$$\mathbb{P}_x\left[\|\mathcal{P}_{f,N}(\mu_{N,x}^{t_u+1})\|_1^2 > \frac{\delta^2}{4}\right] \le \frac{4\mathbb{E}[\|\mathcal{P}_{f,N}(\mu_{N,x}^{t_u+1})\|_1^2]}{\delta^2} \le C_\varepsilon\frac{4B_{\varepsilon,\varepsilon_1}(n)}{\delta^2}.$$

For $N$ sufficiently large, this is at most $\varepsilon_0$. □

*Proof of Theorem 2.13.* Fix $\varepsilon_0, \varepsilon_1, \delta > 0$. We wish to show that

$$(1-\varepsilon_1)\log_k(n) \leq t_{\text{mix}}(\delta, N) \leq (1+\varepsilon_1)\log_k(n),$$

except at possibly $\varepsilon_0 n$ starting points of the walk, as long as $n$ is large enough. For the lower bound on the mixing time, we note that the time $t+1$ distribution of the walk is supported on the sphere $S_x^{t+M}$ of radius $t+M$ centered at $x$, which has size at most $k^{t+M}$. Thus,

$$\begin{aligned} d_x(t+1, N) &= \sum_y \left| \mu_{N,x}^{t+1}(y) - \frac{1}{n} \right| \\ &\geq \sum_{y \notin S_x^{t+M}} \left| \mu_{N,x}^{t+1}(y) - \frac{1}{n} \right| \\ &= \frac{n - |S_x^{t+M}|}{n} \\ &\geq 1 - \frac{k^{t+M}}{n}. \end{aligned}$$

This is at least $\delta$ when $t \leq \log_k(n) - \log_k(\frac{1}{1-\delta}) - M$, so that $t_{\text{mix}}(\delta, N) \geq (1-\varepsilon_1)\log_k(n)$. For the upper bound, we choose $N$ and $x$ satisfying the conclusion of Proposition 3.11, and so that $t_u \geq t_\infty$ as in Proposition 3.9. Then $d_x(t_u, N) \leq \delta$, so that $t_{\text{mix}}(\delta, N) \leq (1+\varepsilon_1)\log_k(n)$ for such starting points $x$. □

# Chapter 4

# Walks on $\mathrm{PGL}_d$

The aim of this chapter is to prove Theorem 2.14. Throughout, we set $\mathbf{G} = \mathrm{PGL}_d$.

## 4.1 Geometry of $\mathcal{B}$

Fix the maximal compact subgroup $K = \mathrm{PGL}_d(\mathcal{O})$ in $G = \mathbf{G}(F)$. The vertices of the building $\mathcal{B}(\mathrm{PGL}_d(F))$ are cosets $G/K$, which we may identify with homothety classes of lattices by $gK \sim g\mathcal{O}^d$. Write $\xi$ for the vertex corresponding to the homothety class of $\mathcal{O}^d$, i.e. to the coset $K$.

A collection $g_0\xi, \ldots, g_j\xi$ of vertices forms a $j$-cell if there exist representative lattices such that $\varpi L_0 < L_j < \ldots < L_1 < L_0$, reordering if necessary.

The vertices are colored

$$\mathrm{col}(gK) = \mathrm{ord}(\det(g)) \pmod d.$$

For $j \geq 1$, we say an ordered $j$-cell $\sigma$ is of *type one* if $\mathrm{col}(\sigma_{i+1}) = \mathrm{col}(\sigma_i) + 1 \mod d$ for $0 \leq i < j$, and we denote the set of type one $j$-cells by $\mathcal{B}_1^j$. Notice that while the color of a vertex is not $G$-invariant, differences between colors are. Thus, for a quotient $X = \Gamma \setminus \mathcal{B}$, we may speak of $X_1^j = \Gamma \setminus \mathcal{B}_1^j$.

For $\alpha = (\alpha_1, \ldots, \alpha_d)$, we write $\varpi^\alpha = \mathrm{diag}(\varpi^{\alpha_1}, \ldots, \varpi^{\alpha_d})$. The *fundamental apartment* $\mathcal{A}$ of $\mathcal{B}$ is

the subcomplex induced by translations of $\xi$ by diagonal matrices. In particular, a vertex $\varpi^\alpha\xi$ is adjacent to $\varpi^{\alpha+\gamma}\xi$ for $\gamma$ a non-constant binary vector. The spherical Weyl group $S_d$ acts on $\mathcal{A}$, for which a fundamental domain is given by the sector $\mathcal{S} = A^+\xi$ where

$$A^+ = \{\varpi^\alpha \mid \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{d-1} \geq \alpha_d = 0\}.$$

We may identify $\mathcal{S}$ with $\mathbb{N}^{d-1}$ by $\varpi^\alpha\xi \mapsto (\alpha_1 - \alpha_2, \ldots, \alpha_{d-1} - \alpha_d)$, giving it the structure of a simplicial complex, and define the $R$-norm of $\vec{x} \in \mathbb{N}^{d-1}$ by

$$R(\vec{x}) = \sum_{j=1}^{d-1} j(d-j)x_j.$$

By the Cartan decomposition, we may also identify $\mathcal{S} \cong K \setminus \mathcal{B}$, and denote by $\Phi$ the projection $\mathcal{B} \to \mathbb{N}^{d-1}$.

## 4.2 Double Coset Operators

Let $\sigma_1$ and $\sigma_2$ be cells in $\mathcal{B}$ with $G$-stabilizers $G_{\sigma_1}$ and $G_{\sigma_2}$. Then for any double coset $G_{\sigma_1}gG_{\sigma_2}$, we may define a $G$-equivariant branching operator from $G\sigma_1$ to $G\sigma_2$ by

$$(G_{\sigma_1}gG_{\sigma_2})(g'\sigma_1) = g'G_{\sigma_1}g\sigma_2.$$

Of particular interest to us are the *geodesic flow operators* $T_j$ on $\mathcal{B}_1^j$, which send the cell $[v_0, \ldots, v_j]$ to each type one cell $[v_1, \ldots, v_j, w]$ for which $\{v_0, \ldots, v_j, w\} \notin \mathcal{B}$.

*Remark* 4.1. We recall that the analysis in §3.1 is of branching operators defined on sets $\mathfrak{C}$ of unordered cells, while $\mathcal{B}_1^j$ consists of ordered cells. When $j < d-1$, the type one condition specifies a unique ordering, but for $j = d-1$ the ordering is not unique. To accommodate the ordering, one replaces the setwise stabilizers with pointwise stabilizers, and the proofs proceed similarly. In particular, Proposition 3.3 and Theorem 3.6 still apply.

In addition, for each $0 \leq j \leq d-2$, we have the operator $T_{j,j+1}$ sending $\sigma \in \mathcal{B}_1^j$ to each $[v, \sigma] \in \mathcal{B}_1^{j+1}$,

as well as $T_{d-1,0}$ which takes $\sigma \in \mathcal{B}_1^{d-1}$ and returns its last vertex.

**Proposition 4.2.** *For $\vec{x} \in \mathbb{N}^{d-1}$, the fiber $\Phi^{-1}(\vec{x}) = T_{\vec{x}}(\xi)$ where*

$$T_{\vec{x}} = T_{d-1,0} \prod_{j=d-1}^{1} T_j^{jx_j} T_{j-1,j} : \mathcal{B}^0 \to \mathcal{B}^0.$$

*In particular, we may bound the size of the fiber by*

$$|\Phi^{-1}(\vec{x})| \leq d!\ q^{\binom{d}{2}+R(\vec{x})}.$$

*Proof.* See [CP22, Proposition 4.3 and Corollary 4.5]. □

*Remark* 4.3. We notice the product is written in descending order. For instance, when $d = 3$ we have:

$$T_{(x,y)} = T_{2,0}\ T_2^{2y}\ T_{1,2}\ T_1^x\ T_{0,1}.$$

## 4.3 Projected random walks on $\mathcal{S}$

Given a SRW on $\mathcal{B}$, we consider the projected random walk $\vec{x}(t) = (x_1(t), \ldots, x_{d-1}(t))$ on $\mathbb{N}^{d-1}$, and define $\rho(t) = R(\vec{x}(t))$. Away from the boundary, for a non-constant binary vector $\gamma$ we have

$$\mathbb{P}[\varpi^\alpha \xi \to \varpi^{\alpha+\gamma}\xi] = \frac{q^{Z_\gamma}}{\deg(\xi)},$$

where $Z_\gamma$ counts pairs $i < j$ for which $\gamma_i = 1$ and $\gamma_j = 0$. At the boundary, any $\gamma$ for which $\alpha + \gamma$ is outside $\mathcal{S}$ is folded back by the action of the spherical Weyl group.

**Proposition 4.4.** *[CP22, Proposition 4.2] The projected walk visits $\partial\mathcal{S}$ only finitely many times with probability* 1*.*

When the walk is away from the boundary, the distribution $\mathscr{D}$ of the change in the $R$-distance is

$$\mathbb{P}[\mathscr{D} = j] = \sum_{\gamma : R(\gamma') = j} \frac{q^{Z_\gamma}}{\deg(\xi)},$$

where $\gamma$ runs over non-constant binary $d$-tuples, and $\gamma' = (\gamma_1 - \gamma_2, \ldots, \gamma_{d-1} - \gamma_d)$ is the associated difference vector. Define $\mathcal{E}_d = \mathbb{E}[\mathscr{D}]$ and $\sigma_d = \sqrt{\mathrm{Var}[\mathscr{D}]}$.

**Proposition 4.5.** *[CP22, Proposition 4.7] There exists some $s = s(q, \delta)$ such that the $1 - 3\delta$ mixing time of SRW on sufficiently large quotients $\Gamma \setminus \mathcal{B}$ is greater than*

$$t_0 = \frac{1}{\mathcal{E}_d} \log_q(n) - (s+1)\sqrt{\log_q(n)}.$$

*Remark* 4.6. While Chapman and Parzanchevski restrict attention to Ramanujan complexes, the lower bound on the mixing time does not rely on this assumption.

Now we consider a family of lattices $\{\Gamma_N\}$ satisfying the density hypothesis, and the associated family of quotients $X_N = \Gamma_N \setminus \mathcal{B}$. There exists a unique $m$ dividing $d$ such that

$$\{\mathrm{ord}_\varpi(\det(\gamma)) \mid \gamma \in \Gamma\} = m\mathbb{Z},$$

so the complexes are $m$-partite, and we have a new coloring map $\mathrm{col} : X_N^0 \to \mathbb{Z}/m\mathbb{Z}$, regarded as a simplicial map from $X_N$ to the $(m-1)$-dimensional simplex $\Delta_{m-1}$. Denoting by $\mu_{N,v}^t$ the distribution of the walk on $\mathfrak{X}_N$ at time $t$ starting at $v$, we estimate

$$\|\mu_{N,v}^t - u\|_1 \leq \|\mathcal{P}_{0,N}(\mu_{N,v}^t)\|_1 + \|\mathcal{P}_{\mathrm{col},N}(\mu_{N,v}^t) - u\|_1,$$

where $\mathcal{P}_{\mathrm{col},N}$ is the projection to the space $L^2_{\mathrm{col}}(X_N^0)$ of functions which are constant on each color, and $\mathcal{P}_{0,N}$ is the projection to the orthogonal complement. The second term is controlled by a walk on $\Delta_{m-1}$ for which every edge is taken with positive probability. If $d = m = 2$, the complexes $X_N$ are bipartite graphs, in which case we modify the walk to be lazy in its first step to make it aperiodic. Otherwise, SRW induces an aperiodic walk on $\Delta_{m-1}$: if $m = 2$ and $d > 2$, then there are loops at the vertices. Thus, for any $\delta$, there is some $t_\Delta$ (independent of $N$) such that if $t \geq t_\Delta$, $\|\mathcal{P}_{\mathrm{col},N}(\mu_{N,v}^t) - u\|_1 \leq \delta$ for all $v \in X_N^0$.

Now we move to bound $\|\mathcal{P}_{0,N}(\mu_{N,v}^t)\|_1$. Choose the covering map $\varphi : \mathcal{B} \to X_N$ such that $\varphi(\xi) = v$. Write $\mu_{N,v}^{t,\vec{x}}$ to be the distribution at time $t$ conditioned on the projection of the lifted walk to $\mathcal{S}$

satisfying $\vec{x}(t) = \vec{x}$, and define the following quantities for $s, \varepsilon_1 > 0$:

$$r_1 = (1+\varepsilon_1)\log_q(n),$$
$$t_1 = t_1(s) = (1+\varepsilon_1)\mathcal{E}_d^{-1}\log_q(n) + s\sqrt{\log_q(n)}.$$

**Theorem 4.7.** *For any $\varepsilon_1, \delta > 0$, there exists $s = s(q, \varepsilon_1, \delta)$ such that for large enough $n$, we have*

$$\mathbb{P}[\rho(t_1(s)) < r_1] \leq \mathbb{P}[Z < -c_{\varepsilon_1} s] + \delta < 2\delta,$$

*where $Z \sim \mathcal{N}(0,1)$ and $c_{\varepsilon_1} = \frac{\mathcal{E}_d^{3/2}}{\sigma_d\sqrt{1+2\varepsilon_1}}$.*

*Remark* 4.8. Note that this theorem concerns the walk on $\mathcal{B}$ rather than $X_N$. The relevance of the quotient is in the number $n$. The proof is adapted from that of [CP22, Proposition 3.4].

*Proof.* Define

$$\Xi(t) = \frac{\rho(t) - \mathcal{E}_d t}{\sigma_d\sqrt{t}}.$$

If $\rho(t_1) < r_1$, then:

$$-\Xi(t_1) > \frac{\mathcal{E}_d t_1 - r_1}{\sigma_d\sqrt{t_1}} = \frac{s\mathcal{E}_d\sqrt{\log_q n}}{\sigma_d\sqrt{t_1}}.$$

For $n$ large enough we have $s\sqrt{\log_q n} \leq \frac{\varepsilon_1}{\mathcal{E}_d}\log_q n$, and thus:

$$t_1 \leq \frac{1+2\varepsilon_1}{\mathcal{E}_d}\log_q n.$$

From $\sqrt{t_1} \leq \sqrt{\frac{1+2\varepsilon_1}{\mathcal{E}_d}}\sqrt{\log_q n}$, it follows that:

$$\frac{s\mathcal{E}_d\sqrt{\log_q n}}{\sigma_d\sqrt{t_1}} \geq \frac{s\mathcal{E}_d\sqrt{\log_q n}}{\sigma_d\sqrt{\frac{1+2\varepsilon_1}{\mathcal{E}_d}}\sqrt{\log_q n}} = \frac{s\mathcal{E}_d^{3/2}}{\sigma_d\sqrt{1+2\varepsilon_1}} = c_{\varepsilon_1} s.$$

Now, we note that $-\Xi(t_1)$ converges to $-Z$ in distribution (see the discussion preceding [CP22,

Proposition 3.4]) and $t_1 \to \infty$ as $n \to \infty$. Thus, for $n$ large enough we have:

$$|\mathbb{P}[-\Xi(t_1) > c_{\varepsilon_1} s] - \mathbb{P}[-Z > c_{\varepsilon_1} s]| < \delta.$$

Noting that $\mathbb{P}[-Z > c_{\varepsilon_1} s] = \mathbb{P}[Z < -c_{\varepsilon_1} s]$, we conclude that:

$$\mathbb{P}[\rho(t_1) < r_1] \leq \mathbb{P}[-\Xi(t_1) > c_{\varepsilon_1} s] \leq \mathbb{P}[Z < -c_{\varepsilon_1} s] + \delta.$$

For $s$ sufficiently large, we have $\mathbb{P}[Z < -c_{\varepsilon_1} s] < \delta$, completing the proof. □

Fix $\varepsilon_1, \delta > 0$, and let $s = s(q, \varepsilon_1, \delta)$ be the constant provided by Theorem 4.7. For $N$ sufficiently large, we have the bound

$$\begin{aligned} \|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1})\|_1 &= \left\| \mathbb{P}[\rho(t_1) < r_1] \mathcal{P}_{0,N}(\mu_{N,v}^{t_1} \mid \rho(t_1) < r_1) + \sum_{\vec{x}: R(\vec{x}) \geq r_1} \mathbb{P}[\vec{x}(t_1) = \vec{x}] \mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}}) \right\|_1 \\ &\leq 2\delta + \sum_{\vec{x} \in S} \|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_1, \end{aligned}$$

where

$$S = \left\{ \vec{x} \in \mathbb{N}^{d-1} \mid \sum_{i=1}^{d-1} x_i \leq t_1 \text{ and } R(\vec{x}) \geq r_1 \right\}.$$

*Remark* 4.9. Recalling the identification of the sector $\mathcal{S}$ with $\mathbb{N}^{d-1}$, we see that when a step is taken along the walk by a binary vector $\gamma$, the vector $\vec{x}$ evolves by $(\gamma_1 - \gamma_2, \ldots, \gamma_{d-1} - \gamma_d)$. In particular, the sum of the coordinates of $\vec{x}$ increases by at most 1, which is why the maximum is taken over those $\vec{x}$ for which $\sum x_i \leq t_1$.

Now we must show:

**Proposition 4.10.** *Fix $\varepsilon_0, \varepsilon_1, \delta > 0$, and let $s = s(q, \varepsilon_1, \delta)$ be the constant provided by Theorem 4.7. For sufficiently large $N$, all but at most $\varepsilon_0 n$ starting vertices $v \in X_N^0$ satisfy*

$$\sum_{\vec{x} \in S} \|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_1 \leq \delta.$$

*Proof.* By Cauchy-Schwarz, the square of the desired sum is bounded by:

$$\left(\sum_{x\in S}\|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_1\right)^2 \leq |S|\sum_{x\in S}\|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_1^2.$$

Averaging this over all starting vertices $v \in X_N^0$ gives

$$\frac{1}{n}\sum_{v\in X_N^0}\left(\sum_{x\in S}\|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_1\right)^2 \leq |S|\sum_{x\in S}\left(\frac{1}{n}\sum_{v\in X_N^0}\|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_1^2\right).$$

We bound the inner average for fixed $x \in S$. By Cauchy-Schwarz, we have:

$$\frac{1}{n}\sum_{v\in X_N^0}\|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_1^2 \leq \sum_{v\in X_N^0}\|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_2^2.$$

Recall by Proposition 4.2 that for any $t$, we have

$$\mu_{N,v}^{t,\vec{x}} = \widetilde{T_{\vec{x}}}(\mathbb{1}_v) = \widetilde{T_{d-1,0}}\prod_{j=d-1}^{1}\widetilde{T}_j^{\,jx_j}\widetilde{T_{j-1,j}}(\mathbb{1}_v),$$

where $\widetilde{T}_i$ and $\widetilde{T}_{ij}$ are the stochastic operators associated to $T_i$ and $T_{ij}$, respectively, acting on $L^2(X_N^0)$. These operators respect the decompositions $L^2 = L^2_{\text{col}} \oplus L_0^2$ and $L_0^2 = \bigoplus_{p<\infty} V(p,N)$, so that

$$\sum_{v\in X_N^0}\|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_2^2 \leq \sum_{v\in X_N^0}\sum_{p<\infty}\|\widetilde{T}_{\vec{x}}|_{V(p,N)}\|^2\cdot\|\mathcal{P}_{p,N}(\mathbb{1}_v)\|_2^2.$$

To estimate $\|\widetilde{T}_{\vec{x}}|_{V(p,N)}\|$, we first note that the product of the operator norms of the $\widetilde{T}_{ij}$ is at most 1. Since the $T_j$ are $q^{d-j}$-regular, we have by Theorem 3.6 (see also Remark 4.1) that

$$\left\|\widetilde{T}_j^{jx_j}|_{V(p,N)}\right\| \ll (jx_j)^r q^{-(d-j)jx_j/p},$$

and thus

$$\left\|\widetilde{T}_{\vec{x}}|_{V(p,N)}\right\| \ll \left(\prod_{j=d-1}^{1}(jx_j)^r\right)q^{-R(\vec{x})/p}.$$

Substituting this bound into the average, we have

$$\frac{1}{n}\sum_{v\in X_N^0}\|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_1^2 \ll \left(\prod_{j=d-1}^{1}(jx_j)^{2r}\right)\sum_{p<\infty}q^{-2R(\vec{x})/p}\sum_{v\in X_N^0}\|\mathcal{P}_{p,N}(\mathbb{1}_v)\|_2^2.$$

Since each $x_j \le t_1$ and $j \le d$, we may bound the product as

$$\prod_{j=d-1}^{1}(jx_j)^{2r} \le (dt_1)^{2rd}.$$

To bound the sum, we again use integration by parts. Define the quantities:

- $a(p) = \sum_{v\in X_N^0}\|\mathcal{P}_{p,N}(\mathbb{1}_v)\|_2^2$,
- $A(p) = \sum_{p'\in[p,\infty)} a(p')$, and
- $f(p) = q^{-2R(\vec{x})/p}$.

Then

$$\sum_{p<\infty}a(p)f(p) = A(2)f(2) + \int_2^\infty A(p)f'(p)dp.$$

Because the $X_N$ form an expander family, the integral may be truncated at some finite $C$. Thus, the latter expression equals

$$A(2)\ q^{-R(\vec{x})} + \int_2^C\left(\sum_{p'\in[p,\infty)}\sum_{v\in X_N^0}\|\mathcal{P}_{p',N}(\mathbb{1}_v)\|_2^2\right)\frac{2R(\vec{x})\log(q)}{p^2}q^{-2R(\vec{x})/p}dp.$$

The argument for Lemma 3.7 holds verbatim in this setting, so we apply its statement to bound the sum within the integral and use $A(2) \le n$ to see that

$$\sum_{p<\infty}a(p)f(p) \ll_\varepsilon \frac{n}{q^{R(\vec{x})}} + 2R(\vec{x})\log(q)n^\varepsilon\int_2^C n^{2/p}q^{-2R(\vec{x})/p}p^{-2}dp.$$

Applying the change of variables $u = 1/p$, we see that the integral evaluates to

$$\frac{n/q^{R(\vec{x})} - (n/q^{R(\vec{x})})^{2/C}}{2\log(n/q^{R(\vec{x})})} = \frac{(n/q^{R(\vec{x})})^{2/C} - n/q^{R(\vec{x})}}{2\log(q^{R(\vec{x})}/n)}.$$

Recall the assumption that $\vec{x} \in S$, so $R(\vec{x}) \geq r_1 = (1+\varepsilon_1)\log_q(n)$. Thus, $n/q^{R(\vec{x})} \leq n^{-\varepsilon_1}$, $n/q^{R(\vec{x})} \leq (n/q^{R(\vec{x})})^{2/C}$, and $\log(q^{R(\vec{x})}/n) \geq \varepsilon_1 \log(n)$, so we bound the integral by

$$\frac{n^{-2\varepsilon_1/C}}{2\varepsilon_1 \log(n)}.$$

Substituting this gives the following bound for the average over a fixed $x \in S$:

$$\frac{1}{n}\sum_{v\in X_N^0} \|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_1^2 \ll_\varepsilon (d(t_1))^{2rd}\left(n^{-\varepsilon_1} + R(\vec{x})\log(q)\frac{n^{\varepsilon-2\varepsilon_1/C}}{\varepsilon_1\log(n)}\right).$$

We denote the right hand expression by $B_{\varepsilon,\varepsilon_1}(n)$. For $\varepsilon < 2\varepsilon_1/C$, $B_{\varepsilon,\varepsilon_1}$ decays polynomially in $n$. Summing over $x \in S$, we have

$$\frac{1}{n}\sum_{v\in X_N^0}\left(\sum_{x\in S}\|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_1\right)^2 \leq |S|\sum_{x\in S}\left(\frac{1}{n}\sum_{v\in X_N^0}\|\mathcal{P}_{0,N}(\mu_{N,v}^{t_1,\vec{x}})\|_1^2\right) \leq |S|^2 B_{\varepsilon,\varepsilon_1}(n).$$

Since elements $x \in S$ satisfy $\sum x_i \leq t_1$, we have

$$|S| \leq \binom{\lfloor t_1\rfloor + d - 1}{d-1} \leq (t_1+d)^d.$$

In particular, the size of $S$ grows at most logarithmically in $n$, so $|S|^2 B_{\varepsilon,\varepsilon_1}(n)$ still decays polynomially in $n$. Arguing by Markov's inequality as in Proposition 3.11, we have that all but $\varepsilon_0 n$ starting vertices satisfy the desired inequality. □

*Proof of Theorem 2.14.* Combining the above bounds, we see that the $4\delta$ mixing time of SRW on $\Gamma_N \setminus \mathcal{B}$ is at most $t_1$ for sufficiently large $n = |X_N^0|$. Shrinking $\varepsilon_1$ if needed, we get that this mixing time is at most $(1+\varepsilon_1)\mathcal{E}_d^{-1}\log_q(n)$. This, combined with Proposition 4.5, establishes cutoff at time $\mathcal{E}_d^{-1}\log_q(n)$, with window $\varepsilon_1\log_q(n)$. □

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